8. KULIKOVSKII A.G. and SVESHNIKOVA E.I., On shock waves propagating over the state of stress in isotropic non-linearly elastic media, PMM, 44, 3, 1980.
9. KULIKOVSKII A.G. and SVESHNIKOVA E.I., Investigation of the shock adiabatic of quasitransverse shock waves in a prestressed elastic medium, PMM, 46, 5, 1982.
10. BLAND D.R., Non-linear Dynamical Elasticity Theory, Mir, Moscow, 1972.
11. KULIKOVSKII A.G., On equations describing quasitransverse wave propagation in a slightly non-isotropic elastic body, PMM, 50, 4, 1986.
12. SVESHNIKOVA E.I., Simple waves in a non-linearly elastic medium, PMM, 46, 4, 1982.
13. GODUNOV S.K., On the concept of a generalized solution, Dokl. Akad. Nauk SSSR, 134, 6, 1960.
14. GODUNOV S.K., On the non-unique smoothing of discontinuities in solutions of quasilinear systems, Dokl. Akad. Nauk SSSR, 136, 2, 1961.

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A METHOD OF INVESTIGATING WEAKLY NON-LINEAR INTERACTION BETWEEN ONE-DIMENSIONAL WAVES*

A.V. KRYLOV

A method of constructing asymptotic approximations of wide classes of solutions of weakly non-linear systems is proposed based on the averaging scheme developed in /1-3/.**(**See also: Krylov A.V. and Shtaras A.L. Internal averaging of multidimensional weakly non-linear systems along characteristics, Dep. in LitNIINTI, 10.11.86, No.1750, 1986). The method enables one to obtain the conditions for the asymptotic decay of systems described by the Burgers, Korteweg-de Vries and similar scalar equations, and also enables one to investigate problems in which this decay does not occur. As an example we investigate the propagation of perturbations in an elastic non-uniform tube. The interaction between two waves is considered and the conditions for resonance are obtained.

1. Non-linear wave phenomena are usually studied using simplifying assumptions of a heuristic form. Hence, a theoretical justification is necessary as well as an investigating of the limit of suitability of the solutions obtained.

Suppose the solution of the quasilinear system

$$U_t + A(U)U_x = 0,\ U = (u_1, \ldots, u_n),\ A(U) = \| a_{ij}(u_1, \ldots, u_n) \| \tag{1.1}$$

is close $(0 < \varepsilon \ll 1)$ to a certain state of equilibrium $(U_0 \equiv \text{const})$

$$U = U_0 + \varepsilon U_1(t, x, \varepsilon) \tag{1.2}$$

We assume that the constants (const) are everywhere independent of ε; the subscripts i and j take the values $1, 2, \ldots, n$.

If problem (1.1), (1.2) is hyperbolic (/4/, p.23), then by making the replacement $V = RU_1$, $R = \| r_{ij} \|$, $\det R \neq 0$ it can be reduced to the form

$$V_t + \Lambda V_x = -\varepsilon R A_1[R^{-1}V] R^{-1} V_x + o(\varepsilon) \tag{1.3}$$

$$\Lambda \equiv \operatorname{diag}\{\lambda_1, \ldots, \lambda_n\} = RA(U_0)R^{-1}$$

$$A_1[U_1] = \frac{dA(U_0)}{dU} U_1 \equiv \left\| \sum_{k=1}^{n} \left[\frac{\partial}{\partial u_k} a_{ij}(U_0) \right] u_{1k} \right\|$$

The initial condition

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$$U_1(0, x, \varepsilon) = \Phi(x) = (\varphi_1(x), \ldots, \varphi_n(x)) \tag{1.4}$$

specifies the long-wave solution of the problem (the wavelengths $O(1/\lambda_j) = \text{const}$ considerably exceed their amplitude $O(\varepsilon)$). In the linearized theory we confine ourselves to the approximation obtained from (1.3) when $\varepsilon = 0$

$$U_1 = R^{-1}\Big(\sum_{k=1}^{n} r_{1k}\varphi_k(x - \lambda_1 t), \ldots, \sum_{k=1}^{n} r_{nk}\varphi_k(x - \lambda_n t)\Big) \tag{1.5}$$

This approximation describes n non-interacting linear hyperbolic waves (/5/, p.9) travelling with velocities $\lambda_1, \ldots, \lambda_n$.

In the case of one equation $(n = 1)$ the problem has the form

$$v_t + \lambda v_x = \varepsilon f v v_x, \quad v(0, x, \varepsilon) = v_0(x), \quad f = \text{const}$$

and its solution can be represented as the implicit function

$$v(t, x, \varepsilon) = v_0(x - \lambda t + \varepsilon t f v(t, x, \varepsilon)) \tag{1.6}$$

The solution of Eq.(1.6) describes a non-linear wave travelling with velocity λ with a slowly varying profile (as a function of εt). Approximation (1.5) only describes the solution when $\varepsilon t \ll 1$ or $t \ll \varepsilon^{-1}$, although a smooth solution (continuously differentiable) exists for $t \in [0, O(\varepsilon^{-1})]$ (when $t > O(\varepsilon^{-1})$ the solutions are, generally speaking, discontinuous, and are not considered here). Hence, it is of interest to construct asymptotic solutions of problems of the form (1.3) that are suitable when $t \sim \varepsilon^{-1}$.

2. In /1-3/ (and also in the footnote on the previous page) a method is given for the asymptotic integration of weakly non-linear hyperbolic systems with periodic initial conditions. The same idea also enables us to construct the zeroth approximation of wider classes of problems.

Consider the Cauchy problem for a weakly non-linear system

$$\frac{\partial u_j}{\partial t} + \lambda_j \frac{\partial u_j}{\partial x} = \varepsilon f_j\left(t, x, u_1, \ldots, u_n, \frac{\partial u_1}{\partial x}, \ldots, \frac{\partial u_n}{\partial x}, \varepsilon\right) \tag{2.1}$$

$$u_j(0, x, \varepsilon) = u_{0j}(x, \varepsilon), \quad j = 1, 2, \ldots, n, \quad 0 < \varepsilon \ll 1 \tag{2.2}$$

The functions f_j, u_{0j} are fairly smooth, are bounded for all $(t, x) \in \mathbf{R}^2$, and are continuous as the point $\varepsilon = 0$.

Problem (2.1), (2.2) can be set in correspondence to the averaged system

$$\frac{\partial v_j}{\partial \tau} = M_j[f_j], \quad v_j(0, y_j) = u_{0j}(y_j, 0), \quad \tau = \varepsilon t, \quad y_j = x - \lambda_j t \tag{2.3}$$

$$M_j[f_j] \equiv \lim_{T\to\infty} \frac{1}{T}\int_0^T f_j(s, y_j + \lambda_j s, v_1(\tau, y_j + (\lambda_j - \lambda_1)s), \ldots$$
$$v_n(\tau, y_j + (\lambda_j - \lambda_n)s), \ldots, \partial v_k(\tau, y_j + (\lambda_j - \lambda_k)s)/\partial y_k, \ldots, 0)\, ds$$

If the above assumptions are satisfied, then for appropriate classes $\mathbf{M}$ of initial conditions (2.2) the solution of the averaged system (2.3) asymptotically approaches the accurate solution uniformly (as $\varepsilon \to 0$) with respect to $t \in [0, O(\varepsilon^{-1})]$

$$\lim_{\varepsilon\to 0} \max_j \sup_{(t,x)\in[0,\tau_0/\varepsilon]\times\mathbf{R}} |u_j(t, x, \varepsilon) - v_j(\varepsilon t, x - \lambda_j t)| = 0, \tag{2.4}$$
$$\tau_0 = \text{const} > 0$$

The classes $\mathbf{M}$, in particular, may include the following: $\mathbf{C}_\infty^1(\mathbf{R})$ — a set of functions $g(x) \in \mathbf{C}^1(\mathbf{R})$ which possesses the property $\lim g(x) = \bar{g} = \text{const}, \lim dg(x)/dx = 0 \quad (|x| \to \infty)$; $\mathbf{C}_\Lambda^1(\mathbf{R})$ — a set of Λ-periodic functions $g(x) \in \mathbf{C}^1(\mathbf{R})$; $\mathbf{C}^1_{\{\nu_l\}}(\mathbf{R})$ — a set of almost periodic functions $g(x) \in \mathbf{C}^1(\mathbf{R})$ with Fourier indices $\{\nu_l\}$

$$g(x) \sim \sum_{l\in\mathbf{Z}} g_l \exp\{i\nu_l x\}, \quad i = \sqrt{-1}$$

For problem (1.1)-(1.4) the averaged system can be written in the form

$$\frac{\partial v_j}{\partial \tau} = \sum_{k=1}^{n}\sum_{m=1}^{n} f_{jkm} M_j\left[v_k \frac{\partial v_m}{\partial y_m}\right], \quad v_j(0, y_j) = u_{0j}(y_j, 0) \tag{2.5}$$

It follows from the definition of the operator M_j that

$$M_j\left[v_j\frac{\partial v_j}{\partial y_j}\right]\equiv v_j\frac{\partial v_j}{\partial y_j}$$
$$M_j\left[v_i\frac{\partial v_j}{\partial y_j}\right]\equiv M_j[v_i]\frac{\partial v_j}{\partial y_j} \tag{2.6}$$
$$M_j\left[v_j\frac{\partial v_i}{\partial y_i}\right]\equiv 0,\quad i\neq j$$

Hence, for $u_{0j}(x, 0)\in \mathbf{C}_\infty{}^1(\mathbf{R})$ system (2.5) can be split into the scalar equations

$$\frac{\partial v_j}{\partial \tau}=f_{jjj}v_j\frac{\partial v_j}{\partial y_j},\quad v_j(0, y_j)=u_{0j}(y_j, 0) \tag{2.7}$$

Here we have assumed that

$$\langle u_{0j}(x, 0)\rangle\equiv\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}u_{0j}(x, 0)\,dx=0 \tag{2.8}$$

and, moreover, $\lambda_i\neq\lambda_j$ when $i\neq j$. Hence, the solutions of problem (1.3) from the class $\mathbf{C}_\infty{}^1$ can be split into simple waves.

Note that by using the transformation $\bar{v}_j=v_j-\langle u_{0j}\rangle$ the problem can be reduced to the case when (2.8) is satisfied, since $\langle v_j(\tau, x)\rangle=\text{const}=\langle u_{0j}\rangle$. Henceforth, we will assume that limitations of the form (2.8) are also satisfied, although this is not essential.

The asymptotic form (2.7) is identical with the well-known asymptotic reduction method /6-8/. A proof of the decay of the solution into simple waves was obtained in /9, 10/ under the condition that $\Phi(x)$ in (1.4) approaches zero fairly rapidly (exponentially) as $|x\to|\infty$.

For the initial conditions from the classes $\mathbf{C}_\Lambda{}^1$ and $\mathbf{C}^1_{\{v_l\}}$, system (2.5) can be split into scalar Eqs.(2.7) only in special cases. The conditions for the decomposition follow from the properties of the functions of $\mathbf{C}_\Lambda{}^1$ and $\mathbf{C}^1_{\{v_l\}}$ and Eqs.(2.5), (2.6) and (2.8). If $u_{0j}(x)\in\mathbf{C}^1_{\Lambda_j}(\mathbf{R})$, it is sufficient that

$$\begin{gathered}\forall l_k, l_m\in\mathbf{Z}, |l_k|+|l_m|\neq 0:\\ (\lambda_j-\lambda_k)\,l_k/\Lambda_k\neq(\lambda_j-\lambda_m)\,l_m/\Lambda_m,\quad \forall j\neq k\neq m\neq j\end{gathered} \tag{2.9}$$

If $u_{0j}(x)\in\mathbf{C}^1_{\{v_{jl}\}}(\mathbf{R})$ it is sufficient that

$$\begin{gathered}\forall l_k, l_m\in\mathbf{Z}, |v_{kl_k}|+|v_{ml_m}|\neq 0:\\ v_{kl_k}(\lambda_j-\lambda_k)\neq v_{ml_m}(\lambda_j-\lambda_m),\quad \forall j\neq k\neq m\neq j\end{gathered} \tag{2.10}$$

By representing the solutions of the problems in the form of formal Fourier series, we can show that conditions (2.9), (2.10) denote that there is no resonance.

3. The equations of motion of a liquid in an elastic non-uniform tube have the form (/11/, p.120)

$$\rho\,(u_t+uu_x)=-p_x,\ (\rho S)_t+(\rho Su)_x=0,\ p=P(\rho) \tag{3.1}$$

Here u, ρ, p are the values of the velocity, density, and pressure of the liquid respectively, averaged over the transverse cross-section of the tube, and $S(p, x)$ is the cross-section of area of the tube.

Suppose that the liquid at rest at the instant of time $t=0$ receives a weak perturbation

$$u(0, x)=\varepsilon\bar{u}_1(x),\ \rho(0, x)=\rho_0+\varepsilon\bar{\rho}_1(x),\ \rho_0=\text{const}>0,\ 0<\varepsilon\ll 1 \tag{3.2}$$

If, in addition, the ductility of the tube is slightly non-uniform, i.e.

$$S(p, x)=\bar{S}_0(p)+\varepsilon\bar{S}_1(p, x)+\varepsilon^2\bar{S}_2(p, x)+o(\varepsilon^2)$$

or (we take into account (3.1), (3.2))

$$S(p, x)=S_0+\varepsilon\,(A(x)+k\rho_1)+\varepsilon^2\,(C(x)+B(x)\,\rho_1+\eta\rho_1{}^2)+o(\varepsilon^2) \tag{3.3}$$

where (the subscripts ρ and p denote the derivatives with respect to ρ and p, calculated for $\rho=\rho_0,\ p=p_0$)

$$\begin{gathered}S_0=\bar{S}_0(p_0),\ A(x)=\bar{S}_1(p_0, x),\ C(x)=\bar{S}_2(p_0, x),\ p_0=P(\rho_0)\\ B(x)=\bar{S}_{1p}P_\rho,\ k=\bar{S}_{0p}P_\rho,\ \eta={}^1/_2\,[\bar{S}_{0pp}P_\rho{}^2+\bar{S}_{0p}P_{\rho\rho}]\end{gathered} \tag{3.4}$$

then, by substituting the expressions $u = \varepsilon u_1$, $\rho = \rho_0 + \varepsilon\rho_1$ and (3.3) into (3.1), we obtain, with an accuracy $o(\varepsilon)$,

$$\rho_{1t} + \rho_0\gamma u_{1x} = \varepsilon f,\ u_{1t} + P_\rho\rho_0^{-1}\rho_{1x} = \varepsilon g \tag{3.5}$$

$$f = \gamma S_0^{-1}\{\rho_0\gamma u_{1x}[\rho_0(B + 2\eta\rho_1) + A + 2k\rho_1] - S_0(u_1\rho_1)_x - \rho_0[u_1(A + k\rho_1)]_x\} \tag{3.6}$$

$$g = -\{u_1u_{1x} + \rho_1\rho_{1x}\delta\}$$

$$\gamma = \frac{S_0}{S_0 + k\rho_0},\quad \delta = \frac{1}{\rho_0}\left[P_{\rho\rho} - \frac{P_\rho}{\rho_0}\right]$$

By making the change of variables

$$u_1 = \alpha v + w,\ \rho_1 = v - \alpha^{-1}w,\ \alpha = c_1/(\gamma\rho_0),\ c_1 = \sqrt{P_\rho\gamma} \tag{3.7}$$

system (3.5), (3.6) and the initial conditions (3.2) can be reduced to the form

$$\begin{aligned} &v_t + c_1v_x = {}^1/_2\varepsilon(\alpha^{-1}g + f) \equiv \varepsilon F_1 \\ &w_t - c_1w_x = {}^1/_2\varepsilon(g - \alpha f) \equiv \varepsilon F_2 \\ &v(0, x, \varepsilon) = v_0(x) \equiv {}^1/_2[\alpha^{-1}\bar{u}_1(x) + \bar{\rho}_1(x)] \\ &w(0, x, \varepsilon) = w_0(x) \equiv {}^1/_2[\bar{u}_1(x) - \alpha\bar{\rho}_1(x)] \end{aligned} \tag{3.8}$$

The functions

$$\begin{aligned} &F_\varkappa = G_{10}^\varkappa(x)v + G_{20}^\varkappa(x)v_x + G_{01}^\varkappa(x)w + G_{02}^\varkappa(x)w_x + D_{11}^\varkappa vv_x + \\ &\quad D_{12}^\varkappa v_xw + D_{21}^\varkappa vw_x + D_{22}^\varkappa ww_x \\ &\varkappa = 1, 2 \end{aligned}$$

can be found by elementary transformations from (3.6) and (3.7). In particular

$$\begin{aligned} &D_{22}^2 = -{}^1/_2[2 + \gamma - \gamma K\rho_0S_0^{-1} + \alpha^{-2}\delta],\ D_{11}^1 = \alpha D_{22}^2 \\ &G_{01}^1 = -{}^1/_2A_x\rho_0\gamma S_0^{-1},\ G_{02}^1 = {}^1/_2E\rho_0\gamma S_0^{-1} \\ &G_{10}^2 = {}^1/_2A_xP_\rho(\rho_0S_0)^{-1},\ G_{20}^2 = -{}^1/_2P_\rho ES_0^{-1} \\ &K = 2\gamma\rho_0\eta + (2\gamma - 1)k,\ E = E(x) = \gamma\rho_0B(x) + (\gamma - 1)A(x) \end{aligned}$$

If constraints of the form (2.8) are satisfied, namely,

$$\langle v_0(x)\rangle = \langle w_0(x)\rangle = \langle A(x)\rangle = \langle B(x)\rangle = 0$$

then the averaged system for problem (3.8) is

$$\begin{aligned} &\bar{v}_\tau = D_{11}^1\bar{v}\bar{v}_y + \lim_{T\to\infty}\frac{1}{T}\int_0^T\{G_{01}^1(y + c_1s)\bar{w}(\tau, y + 2c_1s) + \\ &\quad G_{02}^1(y + c_1s)\bar{w}_z(\tau, y + 2c_1s)\}\,ds \\ &\bar{w}_\tau = D_{22}^2\bar{w}\bar{w}_z + \lim_{T\to\infty}\frac{1}{T}\int_0^T\{G_{10}^2(z - c_1s)\bar{v}(\tau, z - 2c_1s) + \\ &\quad G_{20}^2(z - c_1s)\bar{v}_y(\tau, z - 2c_1s)\}\,ds \\ &y = x - c_1t,\quad z = x + c_1t \end{aligned} \tag{3.9}$$

Hence, sound waves in an elastic non-uniform tube in the zeroth approximation can be described by the expressions

$$u = \varepsilon[\alpha\bar{v}(\varepsilon t, x - c_1t) + \bar{w}(\varepsilon t, x + c_1t)]$$

$$\rho = \rho_0 + \varepsilon[\bar{v}(\varepsilon t, x - c_1t) - \alpha^{-1}\bar{w}(\varepsilon t, x + c_1t)]$$

where the non-linear waves $\bar{v}$ and $\bar{w}$ are found from system (3.9).

4. If the integral terms in (3.9) equal zero, the solution decomposes into simple waves, described by equations of the form (2.7). This occurs when $v_0, w_0 \in \mathbf{C}_\infty^1$. For the initial data from $\mathbf{C}_\Lambda^1$ or $\mathbf{C}^1_{\{\nu_l\}}$ the conditions for the decomposition are similar to (2.9) and (2.10). Moreover, the decomposition occurs for any initial data in the case of a uniform tube $(S = S(p))$. This can be seen from (3.4) and (3.9).

Suppose $v_0(x),\ w_0(x) \in \mathbf{C}_{2\pi}^1(\mathbf{R})$ and the functions A and B can be represented in the form of Fourier series

$$A(x) = \sum_{l\in\mathbf{Z}} A_l\exp\{i\alpha_lx\} \tag{4.1}$$

$$B(x) = \sum_{l \in \mathbf{Z}} B_l \exp\{i\beta_l x\}$$

If among the Fourier factors α_l, β_l in (4.1) there are even numbers, the conditions for decomposition of the form (2.9), (2.10) are not satisfied and the problem is a resonance problem. Nevertheless, decomposition can occur in this case also.

Integration by parts in (3.9) leads to the sufficient condition for decomposition

$$G_{01}{}^1 = {}^1/{}_2 G^1_{02x}, \quad G_{10}{}^2 = {}^1/{}_2 G^2_{20x}$$

which is equivalent to

$$A_x(\gamma + 1) + B_x \gamma \rho_0 = 0 \tag{4.2}$$

To analyse Eq.(4.2) it is necessary to know the explicit form of the coefficients (3.4) in (3.3). In the case of a circular tube with thin walls the cross-sectional area F can be expressed by the formula (/12/, p.105)

$$\frac{F - F_0}{F_0} = \frac{d}{\delta}\frac{p - p_0}{E} + \frac{1}{4}\left(\frac{d}{\delta}\frac{p - p_0}{E}\right)^2, \quad F_0 = \frac{\pi d^2}{4} \tag{4.3}$$

where d is the internal diameter, δ is the wall thickness, E is the modulus of elasticity of the material of the walls of the tube, and F_0 is the cross-sectional area of the tube when $p = p_0$. The above theory enables one to investigate the behaviour of sound waves in a weakly non-uniform tube

$$d = d_0 + \varepsilon d_1(x), \ \delta = \delta_0 + \varepsilon\delta_1(x), \ E = E_0 + \varepsilon E_1(x) \tag{4.4}$$

From (4.3) and (4.4) we obtain after elementary reduction

$$A(x) = \frac{\pi}{2} d_0 d_1(x_1), \quad B(x) = \frac{\pi P_\rho}{4}\frac{d_0^3}{\delta_0 E_0}\left(\frac{3\,d_1(x)}{d_0} - \frac{E_1(x)}{E_0} - \frac{\delta_1(x)}{\delta_0}\right)$$

and from (4.2) we obtain the condition for decomposition

$$\frac{d_{1x}}{d_0}\left[5 + 4\left(\frac{d_0}{\delta_0 E_0} P_\rho \rho_0\right)^{-1}\right] = \frac{E_{1x}}{E_0} + \frac{\delta_{1x}}{\delta_0}$$

5. If decomposition into simple waves does not occur, problem (3.9) is of independent interest.

Suppose $v_0(x), w_0(x) \in \mathbf{C}_{2\pi}{}^1(\mathbf{R})$, the functions $A(x)$ and $B(x)$ can be represented in the form (4.1), and there are even numbers among the Fourier factors α_l, β_l. Then in system (3.9) the terms calculated by passing to the limit as $T \to \infty$ take the form

$$-\frac{\gamma\rho_0}{4S_0}\sum \frac{\varphi_{lk}}{2\pi}\int_0^{2\pi} \exp\{im(2y + p)\}\,\overline{w}(\tau, y + p)\,dp$$

$$\frac{P_\rho}{4\rho_0 S_0}\sum \frac{\varphi_{lk}}{2\pi}\int_0^{2\pi} \exp\{im(2z - p)\}\,\overline{v}(\tau, z - p)\,dp$$

where $\varphi_{lk} = A_l(\gamma + l) + B_k\gamma\rho_0$; the summation is carried out over l, k: $\alpha_l = \beta_k = 2m \neq 0$.

Suppose, for simplicity, that we retain only three harmonics in (4.1)

$$\left| A(x) - \sum_{j=1}^{3}(a_j{}^s \sin(jx) + a_j{}^c \cos(jx))\right| \ll 1$$

$$\left| B(x) - \sum_{j=1}^{3}(b_j{}^s \sin(jx) + b_j{}^c \cos(jx))\right| \ll 1$$

Then, by making the replacement of variables

$$\overline{v}(\tau, y) = -\alpha^{-1}u^+(\tau, y), \quad \overline{w} = u^-$$

problem (3.9) can be reduced to the form

$$u_\tau{}^\pm \pm eu^\pm u_x{}^\pm = \frac{d}{2\pi}\int_0^{2\pi}\cos(2x - \varphi - s)\,u^\mp(\tau, x - s)\,ds \tag{5.1}$$

$$u^\pm(0, x) = u_0{}^\pm(x) \in \mathbf{C}^1_{2\pi}(\mathbf{R})$$

$$u_0{}^+(x) = -{}^1/{}_2(\bar{u}_1(x) + \alpha\bar{\rho}_1(x)), \ u_0{}^-(x) = w_0(x)$$

$$e = -D_{22}^2, \; d = {}^1/_4 c_1 S_0^{-1} (A_2^2 + B_2^2)$$
$$A_2 = a_2^c (\gamma + 1) + b_2^c \gamma\rho_0, \; B_2 = a_2^s (\gamma + 1) + b_2^s \gamma\rho_0$$
$$\varphi = \arccos [A_2/(A_2^2 + B_2^2)]$$

For the equations of system (5.1), which are close in form to the Whitham equation, well-known in the theory of non-linear waves (/5/, p.459) in the periodic case, one can investigate problems connected with the inversion of the waves /13, 14/. System (5.1) also takes into account non-linear effects, in addition to the inversion just mentioned, as well as changes in the amplitudes of the waves which arise from their resonance interaction. The approximation (u^+, u^-) constructed approximates to the accurate solution of problem (3.8) or (3.1)-(3.3) in the region $(\tau, x) \in \Omega = [0, \tau_0] \times [0, 2\pi]$, $\tau_0 = \mathrm{const} > 0$ in which smooth solutions exist (estimate (2.4) is not justified when $\tau > \tau_0$).

Since ε does not occur explicitly in (5.1) and $\Omega = O(1)$, the problem can be solved by well-known numerical methods /4/, Chapter 3).

When $0 \leqslant \tau \leqslant \tau_0$ inversion of the wave cannot occur, but the initial profiles and amplitudes of the waves are changed considerably.

In Figs.1-4 we show graphs of the solutions of problem (5.1) for $e = 1$, $u_0^+(x) \equiv \sin x$, $\varphi = 0$ (Figs.1-3), and $\varphi = \pi/3$ (Fig.4); the values of the pairs $(d, u_0^-(x))$ in Figs.1-4 are as follows: $(4, 0)$, $({}^1/_2, \sin x)$, $(3, -{}^1/_2 \sin 2x)$, $(4, \sin x)$. The profiles of the waves $u^+(\tau_i, x)$ for $\tau_1 = 0{,}4$ and $\tau_2 = 0{,}8$ are represented by the continuous curves *1* and *2*, and the profiles of the waves $u^-(\tau_i, x)$ are represented by the dashed curves *1* and *2* respectively.

The graphs show how different and complex the behaviour of the solutions of problem (5.1) are compared with the simple waves of the form (2.7).

6. When investigating actual wave processes one often has to take into account the effects of viscosity, heat conduction, and friction, which are expressed by having second and higher derivatives in the equations. The scheme for constructing the averaged system (2.3) can also be applied to such problems.

The equations of plane long waves over a flat bottom have the following form in dimensionless variables (/15/, p.94)

$$\begin{aligned} z_t + (Hu)_x &= \varepsilon \{{}^1/_6 (H^3 u_{xx})_x - {}^1/_2 (Hu)_{xxx} - \\ &\quad HH_x (Hu)_{xx} - (zu)_x\} \\ u_t + z_x &= -\varepsilon u u_x, \quad 0 < \varepsilon \leqslant 1 \end{aligned} \tag{6.1}$$

where $H(x, \varepsilon)$ is the specified dimensionless equation of the bottom.

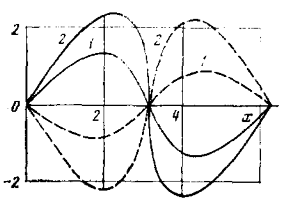

Fig.1

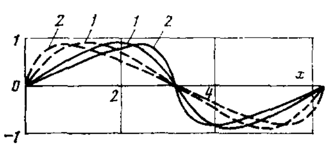

Fig.2

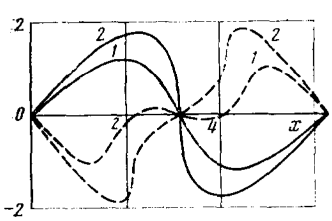

Fig.3

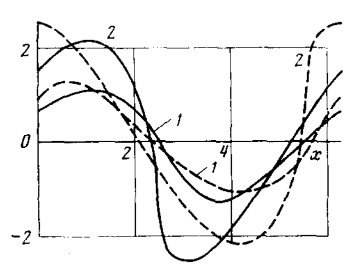

Fig.4

If $H(x, \varepsilon) = 1 + \varepsilon h(x)$, then by making the change of variables $u = v^+ - v^-$, $z = v^+ + v^-$, system (6.1) can be reduced to the form

$$v_t^\pm \pm v_x^\pm = {}^1/_2 \varepsilon (f \mp g) + o(\varepsilon) \tag{6.2}$$
$$f = -{}^1/_3 (v_{xxx}^+ - v_{xxx}^-) - 2 (v_x^+ v^+ - v_x^- v^-) - [h (v^+ - v^-)]_x$$
$$g = (v^+ - v^-)(v_x^+ - v_x^-)$$

The averaged system for (6.2) has the form

$$\frac{\partial}{\partial\tau}\bar{v}^{\pm} \pm \frac{3}{2}\bar{v}^{\pm}\frac{\partial}{\partial y^{\pm}}\bar{v}^{\pm} \pm \frac{1}{6}\frac{\partial^3}{(\partial y^{\pm})^3}\bar{v}^{\pm} = \tag{6.3}$$
$$\pm\frac{1}{2}\lim_{T\to\infty}\frac{1}{T}\int_0^T \frac{\partial}{\partial y^{\pm}}[h(y^{\pm} \pm p)\bar{v}^{\pm}(\tau, y^{\pm} \pm 2p)]\,dp,$$
$$y^{\pm} = x \mp t$$

If, for example, $h = \mathrm{const}$, the right-hand side of (6.3) is zero and the system splits into two scalar Korteweg-de Vries equations, which are identical with those obtained by the reduction method.

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REFERENCES

1. SHTARAS A.L., Asymptotic integration of slightly non-linear partial differential equations, Dokl. Akad. Nauk SSSR, 237, 3, 1977.
2. KRYLOV A.V., Asymptotic integration of first-order hyperbolic systems, Lit. Mat. Sbornik, 23, 4, 1983.
3. KRYLOV A.V., Asymptotic integration of slightly non-linear partial differential systems, Zh. vychisl. Mat. mat. Fiz., 26, 1, 1986.
4. ROZHDESTVENSKII B.L. and YANENKO N.N., Systems of Quasilinear Equations and their Applications in Gas Dynamics, Nauka, Moscow, 1978.
5. WHITHAM J., Linear and Non-linear Waves, Mir, Moscow, 1977.
6. TANIUTI T., Reductive perturbation method and far fields of wave equations, Suppl. Progr. Theor. Phys. 55, 1974.
7. BKHATNAGAR P., Non-linear Waves in One-Dimensional Dispersed Systems, Mir, Moscow, 1983.
8. ENGEL'BREKHT YU.K. and NIGUL U.K., Non-linear Deformation Waves, Nauka, Moscow, 1981.
9. KALYAKIN L.A., Asymptotic decay into simple waves of the solution of a perturbed hyperbolic system of equations, in: Differential Equations with a Small Parameter, UNTs, Sverdlovsk, 1984.
10. KALYAKIN L.A., Long-wave asymptotic form of the solution of a hyperbolic system of equations, Mat. Sbornik, 124, 1, 1984.
11. LIGHTHILL J., Waves in Liquids, Mir, Moscow, 1981.
12. GINZBURG I.P., Applied Hydrodynamics, IIL, Leningrad, 1958.
13. GABOV S.A., On Whitham's equation, Dokl. Akad. Nauk SSSR, 242, 5, 1978.
14. NAUMKIN P.I. and SHISHMAREV I.A., The existence and breakdown of waves described by Whitham's equation, Dokl. Akad. Nauk SSSR, 288, 1, 1986.
15. ALESHKOV YU.Z., Theory of Waves on the Surface of a Heavy Liquid, Izd. LGU, Leningrad, 1981.

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